

## Section 2: Trigonometric equations

### Notes and Examples

In this section you learn how to solve trigonometric equations.

These notes contain subsections on

- [Principal values](#)
- [Solving simple trigonometric equations](#)
- [More complicated examples of trigonometric equations.](#)

### Principal values

There are infinitely many roots to an equation like  $\sin \theta = \frac{1}{2}$ .

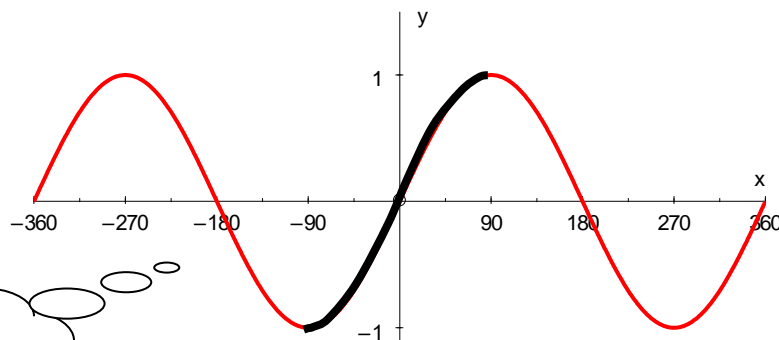
Your calculator will only give one root – the **principal value**.

You find this by pressing the calculator keys for  $\arcsin 0.5$  (or  $\sin^{-1} 0.5$  or  $\text{invsin } 0.5$ ).

Check that you can get the answer of  $30^\circ$ .

You can find other roots by looking at the symmetry of the appropriate graph.

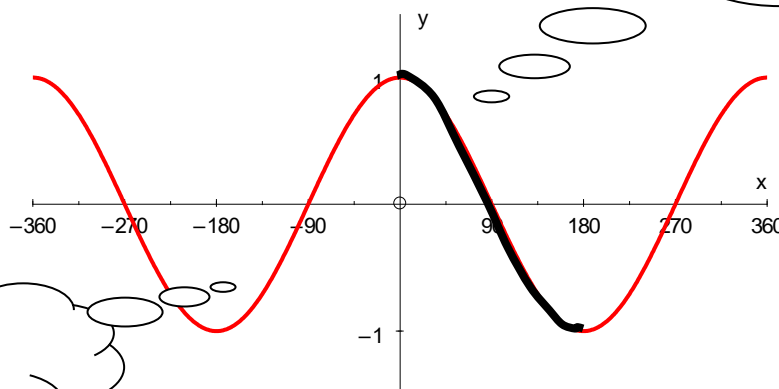
$y = \sin \theta$



A second root in a  $360^\circ$  cycle can be found by  $180^\circ - \theta$

When you use the inverse cosine function, your calculator will always give you an answer from  $0^\circ$  to  $180^\circ$ .

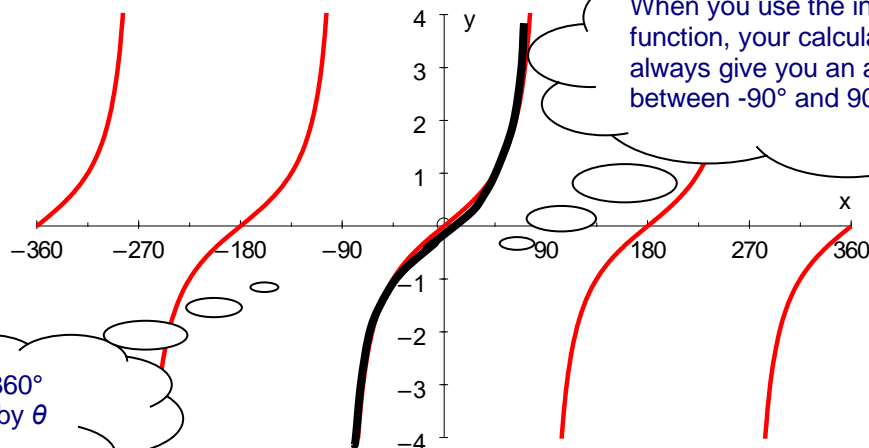
$y = \cos \theta$



A second root in a  $360^\circ$  cycle can be found by  $360^\circ - \theta$

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$$y = \tan \theta$$



Alternatively, you can use the quadrant diagram to find other roots, by thinking about which quadrants the roots will be in.

## Solving simple trigonometric equations

Because there are infinitely many roots to a trigonometric equation you are only ever asked to find a few of them! Any question at this level asking you to solve a trigonometric equation will also give you the interval or range of values in which the roots must lie, e.g. you might be asked to solve  $\tan \theta = 2$  for  $0^\circ \leq \theta \leq 360^\circ$ . You can only directly solve trigonometric equations like  $\sin \theta = \frac{1}{2}$  or  $\cos \theta = \frac{1}{4}$  or  $\tan \theta = -2$ . Here is an example.



### Example 1

Solve  $\sin \theta = \frac{\sqrt{3}}{2}$  for  $-360^\circ \leq \theta \leq 360^\circ$ .

### Solution

$$\sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60^\circ$$

There will be a second root in the second quadrant.

$180^\circ - 60^\circ = 120^\circ$  is also a root.

Since  $y = \sin \theta$  has a period of  $360^\circ$  any other roots can be found by adding/subtracting  $360^\circ$  to these two roots.

So the other roots are:

$$\begin{aligned} 60^\circ - 360^\circ &= -300^\circ \\ \text{and} \quad 120^\circ - 360^\circ &= -240^\circ \end{aligned}$$

So the values of  $\theta$  for which  $\sin \theta = \frac{\sqrt{3}}{2}$  are  $-300^\circ, -240^\circ, 60^\circ, 120^\circ$ .



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Almost all equations of the type in Example 1 have two roots in the range  $0^\circ \leq x < 360^\circ$ .

However, this is not true of all trigonometric equations. Suppose, for example, that you want to solve the equation  $\sin 2x = 0.5$  in the range  $0^\circ \leq x < 360^\circ$ . You can find that  $2x = 30^\circ$  or  $150^\circ$ , which means that  $x = 15^\circ$  or  $75^\circ$ . However, there are two further roots in the range  $0^\circ \leq x < 360^\circ$ , given by  $2x = 30^\circ + 360^\circ = 390^\circ \Rightarrow x = 195^\circ$ , and  $2x = 150^\circ + 360^\circ = 510^\circ \Rightarrow x = 255^\circ$ . So there are four roots:  $x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$ .

This means that if you are solving an equation of the form  $\sin nx = k$ , you need to adjust the range in which you look for initial roots.



## Example 2

Solve each of the following equations in the given range:

- (i)  $\tan 3x = 1$  for  $0 \leq x < 360^\circ$
- (ii)  $\cos(2x + 40^\circ) = 0.5$  for  $-180^\circ < x \leq 180^\circ$

## Solution

(i)  $\tan 3x = 1$   
 $3x = 45^\circ, 225^\circ, 405^\circ, 585^\circ, 765^\circ, 945^\circ$   
 $x = 15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ, 315^\circ$

(ii)  $\cos(2x + 40^\circ) = 0.5$   
 $2x + 40^\circ = -300^\circ, -60^\circ, 60^\circ, 300^\circ$   
 $2x = -340^\circ, -100^\circ, 20^\circ, 260^\circ$   
 $x = -170^\circ, -50^\circ, 10^\circ, 130^\circ$

You need to look for roots for  $3x$  in the range  $0^\circ$  to  $1080^\circ$

The lower end of the range you need to look in is  $-180^\circ \times 2 + 40^\circ = -320^\circ$ , and the upper end is  $180^\circ \times 2 + 40^\circ = 400^\circ$

## More complicated trigonometric equations

Any more complicated equations need to be manipulated algebraically before they can be solved. There are a number of techniques you can use:

1. Rearrange the equation to make  $\cos \theta$ ,  $\sin \theta$  or  $\tan \theta$  the subject.
2. Check to see if the equation factorises to give two (or more) equations which involve just one trigonometric function (see Example 3).  
If it is a quadratic in either  $\sin \theta$ ,  $\cos \theta$ , or  $\tan \theta$  it can either be factorised or solved using the formula for solving quadratic equations (see Example 4).
3. If the equation involves just  $\sin \theta$  and  $\cos \theta$  (and no powers), check to see if you can use the identity  $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$  (see Example 5).
4. If the equation contains a mixture of trigonometric functions (e.g.  $\cos^2 \theta$  and  $\sin \theta$ ) then you may need to use the identity  $\sin^2 \theta + \cos^2 \theta \equiv 1$  to make it a quadratic in either  $\sin \theta$ ,  $\cos \theta$ , or  $\tan \theta$  (see Example 6).

## Example 3



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Solve  $2\cos\theta\sin\theta + \cos\theta = 0$  for  $0^\circ \leq \theta \leq 360^\circ$ .



## Solution

$2\cos\theta\sin\theta + \cos\theta = 0$  can be factorised as there is  $\cos\theta$  in both terms on the LHS.

Factorise:  $\cos\theta(2\sin\theta + 1) = 0$

So either  $\cos\theta = 0$  or  $2\sin\theta + 1 = 0$

$$\cos\theta = 0 \Rightarrow \theta = 90^\circ$$

$360^\circ - 90^\circ = 270^\circ$  is also a root.

$$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2}$$

This has roots in the third and fourth quadrants.

The roots are  $180^\circ + 30^\circ = 210^\circ$  and  $360^\circ - 30^\circ = 330^\circ$ .

So the values of  $\theta$  for which  $2\cos\theta\sin\theta + \cos\theta = 0$  are  $90^\circ$ ,  $210^\circ$ ,  $270^\circ$  and  $330^\circ$ .

It is wrong to divide through by  $\cos\theta$  because you lose the roots to  $\cos\theta = 0$ .

In Example 4 you need to solve a quadratic equation.



## Example 4

Solve  $2\cos^2\theta + 3\cos\theta = 2$  for  $0^\circ \leq \theta \leq 360^\circ$ .

## Solution

$2\cos^2\theta + 3\cos\theta = 2$  is a quadratic equation in  $\cos\theta$

Rearrange the quadratic:  $2\cos^2\theta + 3\cos\theta - 2 = 0$

Let  $\cos\theta = x$ :  $2x^2 + 3x - 2 = 0$

Factorise:  $(2x - 1)(x + 2) = 0$

$$x = \frac{1}{2} \text{ or } x = -2 \Rightarrow \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -2$$

$\cos\theta = -2$  has no real roots.

So we need to solve  $\cos\theta = \frac{1}{2}$

$$\Rightarrow \cos\theta = 60^\circ$$

There is also a root in the 4<sup>th</sup> quadrant, so  $360^\circ - 60^\circ = 300^\circ$  is also a root.

So the values of  $\theta$  for which  $2\cos^2\theta + 3\cos\theta = 2$  are  $60^\circ$  and  $300^\circ$ .

You can replace  $\cos\theta$  with  $x$  to make things simpler! Or factorise straightaway to get:  $(2\cos\theta - 1)(\cos\theta + 2) = 0$  and then solve.

In the next example you need to use the identity  $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ .



## Example 5

Solve  $\sin\theta - 2\cos\theta = 0$  for  $0^\circ \leq \theta \leq 360^\circ$ .

## Solution

You need to rearrange the equation.

$$\sin\theta - 2\cos\theta = 0$$



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Dividing by  $\cos \theta$ :

$$\frac{\sin \theta}{\cos \theta} - 2 = 0$$

Since  $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$ :

$$\tan \theta - 2 = 0$$

$$\Rightarrow \tan \theta = 2$$

$$\Rightarrow \theta = 63.4^\circ \text{ to 1 d.p.}$$

You can safely divide by  $\cos \theta$  because it can't be equal to 0. If it were then  $\sin \theta$  would also have to be 0 and  $\cos \theta$  and  $\sin \theta$  are never both 0 for the same value of  $\theta$ .

There is also a root in the 3<sup>rd</sup> quadrant.  
So  $63.4^\circ + 180^\circ = 243.4^\circ$  is also a root.

So the values of  $\theta$  for which  $\sin \theta - 2 \cos \theta = 0$  are  $63.4^\circ$  and  $243.4^\circ$  to 1 d.p.

In the next example you need to use the trigonometric identity  $\sin^2 \theta + \cos^2 \theta \equiv 1$ .



## Example 6

Solve  $\sin^2 x + \sin x = \cos^2 x$  for  $0^\circ \leq x \leq 360^\circ$

### Solution

Rearranging the identity  
gives:

$$\sin^2 \theta + \cos^2 \theta \equiv 1$$

$$\cos^2 x \equiv 1 - \sin^2 x$$

①

Substituting ① into the equation  $\sin^2 x + \sin x = \cos^2 x$  gives:

$$\sin^2 x + \sin x = 1 - \sin^2 x$$

This is a quadratic in  $\sin x$ .

Rearranging:  $2 \sin^2 x + \sin x - 1 = 0$

Rearranging:  $2 \sin^2 x + \sin x - 1 = 0$

This factorises to give:  $(2 \sin x - 1)(\sin x + 1) = 0$

So either:  $2 \sin x - 1 = 0$  or  $\sin x + 1 = 0$   
 $\Rightarrow \sin x = \frac{1}{2}$   $\Rightarrow \sin x = -1$   
 $\Rightarrow x = 30^\circ \text{ or } 150^\circ$   $\Rightarrow x = 270^\circ$

So the roots to  $\sin^2 x + \sin x = \cos^2 x$  are  $x = 30^\circ, 150^\circ$  or  $270^\circ$

